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ONALMOST WEAK (SEMI WEAK) (SEMI STRONG)CJ-TOPLOGICAL SPACES

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ABSTRACT

In this paper we introduced new types of spaces asalmost weak CJ-space, semi strong CJ-space and semi weak CJ-space. also we studied the relationship between them and the relation of them withalmost CJ-space and almost strong CJ-space, and the relation of them with almost CJ- space and almost strong CJ- space researched by N.A. Dawood and S.G. Gasim..

***Keywords**: almost CJ- space, almost strong CJ-space, almost semi- Strong CJ- space, almost weak CJ-space and almost semi- weak CJ-space.*

I. INTRODUCTION

In [1] E.Michael introduced the concepts of J-space and strong J-space. A topological space X is a J-space if, whenever {A, B} is a closed cover of X with $A \cap B$ compact, then A or B is compact. A space X is a strong J-space if every compact $K \subset X$ is contained in a compact $L \subset X$ with X\L is connected. Michael also introduced three classes of spaces which are closely related to J-space and strong J-space, these spaces are semi strong J-space, weak J-space and semi weak J-space.

In this paper we introduced three types of spaces which are almost semi strong CJ-space, almost weak CJ-space and almost semi weak CJ- space.

Recall that, a space X is said to be compact if every open cover of X contains a finite subcollection that also covers X, and every closed subspace of a compact space is compact, see [2].

A topological space is said to be countably compact if every countable open cover of it has a finite subcover, see [3]. Also in [3] we can see that , a continuous image of a countably compact space is countably compact, and Countably compactness is weakly hereditary property.

Every compact space is countably compact. But the converse is not true, in general, see [4].
A connected space is a topological space X which cannot be represented as the union of two disjoint nonempty open sets. The continuous image of a connected space is connected, see [5].

A perfect map $f : X \to Y$ is a closed, continuous and onto map with $f^{-1}(y)$ compact in X for every $y \in Y$, see [6].

A map $f : X \to Y$ is called boundary - perfect if it is closed and if $\partial f^{-1}(y)$ is compact for every $y \in Y$, see [1].
All maps in this paper are continuous, and all spaces are assumed Hausdorff.

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II. ALMOST SEMISTRONG CJ- SPACE, ALMOSTWEAK CJ-SPACEAND ALMOST SEMI WEAK CJ- SPACE

Definition 2.1: A topological space$(X, \tau)$is an almostsemi strong CJ- space if every compact K⊂X contained ina countably compact $L \subset X$ such that $L \cup C = X$ for some connected $C \subset X\backslash K$.

Definition 2.2: A topological space $(X, \tau)$is said to be an almostweak CJ-space if, whenever $\{A, B, K\}$is a closed coverof X with Kcompact and $A \cap B = \emptyset$, then A or B is countably compact.

Definition 2.3: A topological space $(X, \tau)$is said to be an almostsemi weak CJ-space if, whenever Aand B are disjoint closed subsets of X with compact boundaries, then A or B is countably compact.

Synchronizing with this research we are studying another research, which includes two definitions, almost CJ-space and almost strong CJ- space, we will examine the relationship of our definitions of these two definitions and their relationship with each other. We start by mentioning the two definitions.

Definition 2.4: A space X is said to be almost CJ- spaces if, whenever {A, B} is a closed cover of X with A⋂B compact, then A or B is countably compact.

Definition 2.5: A space X is said to be almost strong CJ- space if every compact K⊂ X is contained in a countably compact L⊂ X such that X\L is connected.

Lemma2.6: If B is a closed non- countably compact subset of any topological space X and C⊂ B is compact, then there is a closed non- countably compact D⊂ B with D⋂C = Ø.

Proof: Let $\mathfrak{A}$ be a countably open cover of B with no finite subcover , and let C ⊂ B be a compact, then $\mathfrak{A}$ is an open cover of C . Pick a finite $\mathfrak{J} \subset \mathfrak{A}$ covering C. Then $D = B \setminus \bigcup \mathfrak{J}$is a closed non–countably compact subset of B with D ∩ C = Ø.

Theorem 2.7: Let X be any topological space , then the following conditions are equivalent:

1. X is an almost CJ-space,
2. For any A⊂X with compact boundary , cl(A) or cl(X\A) is countably compact,
3. If A and B are disjoint closed subsets of X with ∂A or ∂B compact , then A or B is countably compact,
4. If K⊂X is compact , and if $\mathcal{W}$ is a disjoint open cover of X\K ,then X\W is countably compact for some $W \in \mathcal{W}$,
5. Same as (4), but with card $\mathcal{W} = 2$.

Proof:
(1) ⟹ (2): Let A⊂X such that ∂A is compact. Note that {cl(A),cl(X\A)} is a closed cover of X with ∂A=cl(A)⋂cl(X\A) is compact , so cl(A) or cl(X\A) is countably compact by definition of almost CJ-space.

(2)⟹ (3): Let A and B be disjoint closed subsets of X and suppose that ∂A is compact , it follows by (2) that cl(A) or cl(X\A) is countably compact . But cl(A) =A , and B is a closed subset of cl(X\A) , so A or B is countably compact.

(3) ⟹ (1): Let {A, B} be a closed cover of X with $A\cap B$ is compact, we have to show that A or B is countably compact . Suppose that B is non-countably compact and since $A\cap B\subset B$ is compact, so by lemma (2.6) there is a closed non- countably compact $D\subset B$ such that $D\cap(A\cap B)=\emptyset$, it follows that $D\cap A=\emptyset$. Thus A and D are disjoint closed subsets of X and ∂A is a closed subset of $A\cap B$ and thus compact. By (3) A or D is countably compact, but D is non- countably compact. Hence A must be countably compact.

(4) ⟹ (5): Clear.

(5) ⟹ (4): Let $K\subset X$ be compact and let $\mathcal{W}$ be a disjoint open cover of X \ K. To show that X\W is countably compact for some $W\in\mathcal{W}$ we shall follow three demarches.

First, we prove that if U is open subset of X containing K, then $\mathcal{W}' = \{W\in\mathcal{W}: W \not\subseteq U\}$ is finite. Suppose that it is not finite, then $\mathcal{W} = W_1 \cup W_2$ with $W_1 \cap W_2 = \emptyset$ and $W_1 \cap \mathcal{W}'$ and $W_2 \cap \mathcal{W}'$ both finite.

Let $V_1 = \cup W_1$ and $V_2 = \cup W_2$, then $\{V_1, V_2\}$ is a disjoint open cover of X\K , so by (5) $X\backslash V_1$ or $X\backslash V_2$ is countably compact , but $V_1 \subseteq X\backslash V_2$ and $V_2 \subseteq X\backslash V_1$ since V_1 and V_2 are disjoint. It follows that $cl(V_1) \subseteq cl\,(X\backslash V_2) = X\backslash V_2$ and $cl(V_2) \subseteq cl\,(X\backslash V_1) = X\backslash V_1$, so we get $cl(V_1)$ or $cl(V_2)$ is countably compact (since closed subset of countably compact is countably compact).

Suppose that $cl(V_1)$ is countably compact, then $C = cl(V_1)\backslash U$ is countably compact. Now let $\mathcal{W}'_1 = W_1 \cap \mathcal{W}'$, then $\mathcal{W}_1'$ covers C and each $W\in\mathcal{W}'$ intersects C, so C is not countably compact since $\mathcal{W}_1'$ is infinite and disjoint , which is a contradiction. Hence $\mathcal{W}'$ is finite.

Second , we show that if cl(W) is countably compact for all $W\in\mathcal{W}$, then X is countably compact. Let V be a countably open cover of X , then V is a countably open cover of K, which is compact , so V has a finite subcover $\mathcal{F}$ covers K. Let $U = \bigcup\mathcal{F}$, by step one we get a finite family $\mathcal{W}' = \{W\in\mathcal{W}: W \not\subseteq U\}$, so $\bigcup\{cl(W): W\in\mathcal{W}'\}$ is countably compact and since V is a countably open cover of it therefore it is covered by some finite $\mathcal{E}\subset V$. But $\bigcup\mathcal{E}\subset V$ is finite and covers X, so X is countably compact.

Finally,let us show that X\W is countably compact for some $W\in\mathcal{W}$.If cl(W) is countably compact for all $W\in\mathcal{W}$, then X is countably compact by step (2) and since X\W is a closed subset of X , so X\W is countably compact. Suppose that there exists $W_0 \in \mathcal{W}$ such that $cl(W_0)$ is not countably compact.
Let $W^* = \cup\{W\in\mathcal{W}: W \neq W_0\}$, then, $\{W_0, W^*\}$ is a disjoint open cover of $X\backslash K$,so $X\backslash W_0$ or $X\backslash W^*$ is countably compact, by (5). If $X\backslash W^*$ is countably compact, and since $cl(W_0)$ is a closed subset of $X\backslash W^*$, so $cl(W_0)$ is countably compact which is a contradiction, so $X\backslash W^*$ is not countably compact , it follows that $X\backslash W_0$ is countably compact .

(5)⟹ (1): Let {A,B} be a closed cover of X with $A\cap B$ compact , then {X\A,X\B} is a disjoint open cover of $X\backslash A\cap B$, then by (5) X\(X\A) or X\(X\B) is countably compact , that is A or B is countably compact . Hence X is CJ-space.

(1)⟹ (5): Let K⊂X be compact and let $\{W_1, W_2\}$ be a disjoint open cover of X\K, then $\{X\backslash W_1, X\backslash W_2\}$ is a closed cover of X with $X\backslash W_1 \cap X\backslash W_2 = X\backslash(W_1 \cup W_2)$ compact, since $X\backslash K \subset W_1 \cup W_2$, and so $X\backslash(W_1 \cup W_2) \subset K$ which is compact and by the closed subset of compact space is compact. But X is almost CJ-space, so $X\backslash W_1$ or $X\backslash W_2$ is countably compact.

Theorem 2.8: Consider the following properties of a topological space (X, τ),

a) X is an almost strong CJ- space.
b) X is an almostsemi strong CJ- space.
c) X is an almost CJ- space.
d) X is an almost semi weak CJ- space.
e) X is an almost weak CJ-space.

Then (a)⟹(b)⟹(c)⟹(d)⟹(e)

Proof: (a)⟹(b)
Suppose that X is an almost strong CJ- space and let K⊂X be compact, then there exists a countably compact subset L of X such that K⊂L and X\L is connected by definition of almost strong CJ-space. Pick C= X\L, then C is connected and C⊂ X\K since K⊂ L, and C⋃L= X. Hence X is an almostsemistrong CJ- space.

(b) ⟹ (c)

Let X be an almost semi strong CJ- space and let $\{A, B\}$ be a closed cover of X with $A\cap B$ compact, so there exists a countably compact $L \subset X$ such that $A\cap B \subset L$ and there exists a connected subset C of X with $C \subset X\backslash A\cap B$ and $C\cup L = X$ by definition of almost semi strong CJ- space. Note that

$(A\cap C) \cap (B\cap C) = (A\cap B) \cap C = \emptyset$ since $C \subset X\backslash A\cap B$, and that

$(A\cap C) \cup (B\cap C) = (A\cup B) \cap C = X\cap C = C$, so we get a disjoint closed cover $\{A\cap C, B\cap C\}$ of C which is connected, therefor C must be in A⋂C or in B⋂C, then $C\cap B = \emptyset$ or $C\cap A = \emptyset$, it follows that

$B \subset X\backslash C \subset L$ or $A \subset X\backslash C \subset L$ which is countably compact, so A or B is countably compact.

(c) ⟹ (d)
Suppose X is an almost CJ- space and let A, B be two disjoint closed subsets of X with compact boundaries, then A or B is countably compact by Theorem (1.7) Thus X is almost semi weak CJ-space.

(d) ⟹ (e)
Assume that X is an almostsemi weak CJ- space and let $\{A, B, K\}$ be a closed cover of X with K compact and $A\cap B = \emptyset$. But ∂A and ∂B are closed subsets of K since $A^c = B \cup (K\backslash K \cap A)$ and $\partial A = \partial A^c$, so $\partial A = \partial\big(B \cup (K\backslash K \cap A)\big)$, so ∂A⊂ K⋂A⊂ K, similarly we can prove that ∂B⊂ K, and thus ∂A and ∂B are compact, it follows by (d) that A or B is countably compact. Hence X is almost weak CJ- space.

Theorem 2.9: A locally compact space is an almost weak CJ- space if and only if it is an almost CJ-space.

Proof: By Theorem (1.8) every almost CJ- space is an almost weak CJ-space, suppose, then, X is a locally compact almost weak CJ- space, and let $\{A, B\}$ be a closed cover of X. But X is locally compact so $A \cap B \subset \mathrm{Int}(k)$, for some compact $K \subset X$. Let $A^* = A \setminus \mathrm{Int}(K)$ and $B^* = B \setminus \mathrm{Int}(K)$, then $\{A^*, B^*, K\}$ is a closed cover of X with K compact and $A^* \cap B^* = \emptyset$, it follows by definition of almost weak CJ- space, that A^* or B^* is countably compact, then $A^* \cup K$ or $B^* \cup K$ is countably compact since K is compact, and thus countably compact. But A and B are closed subsets of $A^* \cup K$ and $B^* \cup K$ respectively, so A or B is countably compact. Hence X is an almost CJ- space.

Proposition 2.10: If X is an almost CJ- space and $Z = X \cup \{z_\circ\}$, then Z is an almost semi weak CJ- space.

Proof: Let A, B be two disjoint closed subsets of Z with compact boundaries, then $z_\circ \notin A$ or $z_\circ \notin B$. Suppose that $z_\circ \notin B$ and let E= cl(X\B), then {B,E} is a closed cover of X with E$\cap$ B= ∂B which is compact, so B or E is countably compact since X is almost CJ- space. But A$\subset$ E$\cup\{z_\circ\}$, so A or B is countably compact, and thus X is an almost semi weak CJ- space.

Proposition 2.11: Let $\{X_1, X_2\}$ be a closed cover of a topological space X with $X_1 \cap X_2$ non- countably compact. If X_1 and X_2 are almostweak CJ- spaces, then so is X.

Proof: Let $\{A, B, K\}$ be a closed cover of Xwith $A \cap B = \emptyset$ and K is compact. To prove Aor B is countably compact, let $A_i = A \cap X_i$ and $B_i = B \cap X_i$ and $K_i = K \cap X_i$, for i=1,2. Then $\{A_i, B_i, K_i\}$ is a closed cover of X_i with $A_i \cap B_i = \emptyset$ and K_i is countably compact. Now by using the fact saying that X_1 is almost weak CJ- space, we get A_1 or B_1 is countably compact. Suppose that B_1 is countably compact, we claim that B_2 is also countably compact, for if B_2 is not countably compact, so A_2 must be countably compact since X_2 is an almost weak CJ- space, it follows that $C = A_2 \cup B_1 \cup K$ is countably compact, but $X_1 \cap X_2$ is a closed subset of C, so $X_1 \cap X_2$ must be countably compact which is a contradiction. Thus $B = B_1 \cup B_2$ is countably compact. Similarly we can prove that A is countably compact whenever A_1 is countably compact.

Proposition 2.12: Let $\{X_1, X_2\}$ be a closed cover of a topological space X with $X_1 \cap X_2$ non- countably compact. If X_1 and X_2 are almost semi strong CJ- spaces, then so is X.

Proof: Let K$\subset$ X be a compact and let $K_i = K \cap X_i$, then K_i is a closed subset of K, and thus compact subset of the almost semi strong CJ- space X_i, so there exists a countably compact subset L_i of X_i such that $K_i \subset L_i$ and there exists a connected subset C_i of X_i such that $C_i \subset X_i \setminus K_i$ and $L_i \cup C_i = X_i$ (for i=1, 2) by definition of almost semi strong CJ- space. Now let $L = L_1 \cup L_2$ and $C = C_1 \cup C_2$, so L is a countably compact subset of X with K$\subset$ L and C$\cup$L=X and C$\subset$ X\K. It remains to show that C is connected, we need only cheek that $C_1 \cap C_2 \neq \emptyset$ since C_1 and C_2 are connected. Note that $X_1 \cap X_2 \setminus L \neq \emptyset$, for if $X_1 \cap X_2 \setminus L = \emptyset$, then $X_1 \cap X_2$ is a closed subset of L which is countably compact, so $X_1 \cap X_2$ is countably compact which is a contradiction. Also we have $X_i \setminus L \subseteq X_i \setminus L_i \subseteq C_i$, so $(X_1 \cap X_2) \setminus L \subseteq C_1 \cap C_2$, and thus $C_1 \cap C_2 \neq \emptyset$. Hence $C = C_1 \cup C_2$ is connected. Therefor X is an almost semi strong CJ- space.

Proposition 2.13: Let $f: X \to Y$ be an injective perfect map onto Y. Then, if X is an almost semi strong CJ-space, so is Y.

Proof: Let $K \subset Y$ be compact, then $K' = f^{-1}(K)$ is a compact subset of X since f is perfect. But X is an almost semi strong CJ- space, so there exists a countably compact $L' \subset X$ such that $K' \subset L'$ and a connected $C' \subset X \backslash K'$ with $C' \cup L' = X$ by definition of almost semi strong CJ- space. Now let $L = f(L')$ and $C = f(C')$, then L is countably compact and C is connected since f is continuous, moreover $K \subset L$ since f is surjective and $C \subset Y \backslash K$ since f is injective and clear that $L \cup C = Y$. Hence Y is an almost semi strong CJ-space.

Theorem 2.14: The following properties of a space X are equivalent:

a) X is an almostsemi weak CJ-space.
b) If $f: X \to Y$ is boundary- perfect, then $f^{-1}(y)$ is non- countably compact for at most one y$\in$ Y.

Proof:(a) ⟹(b)

Suppose that X is an almost semi weak CJ-space and $y_1 \neq y_2$ in Y, and let $A_i = f^{-1}(y_i)$ (for i=1, 2). Then A_1 and A_2 are closed subsets of X with $A_1 \cap A_2 = \emptyset$ and $\partial A_1, \partial A_2$ are compact since f is boundary- perfect, so A_1 or A_2 is countably compact by definition of almost semi weak CJ- space.

(b) ⟹(a)

Suppose A_1 and A_2 are disjoint closed subsets of X with compact boundaries. Define a relation R on X such that $x \, R \, y \Leftrightarrow x, y \in A_1$ or $x, y \in A_2$.

$$\text{Then } [x] = \left\{\begin{matrix} A_1 & \text{if } x \in A_1 \\ A_2 & \text{if } x \in A_2 \\ \{x\} & \text{if } x \notin A_1 \text{ and } x \notin A_2 \end{matrix}\right\}.$$

Let Y be the quotient space of X with respect to the relation R, and let $f: X \to Y$ be the quotient map, so f is a closed, continuous and onto map. Now to show that f is boundary- perfect, it is sufficient to prove that

$\partial(f^{-1}(y))$ is compact for each $y \in Y$. Let $y \in Y$, then

$$f^{-1}(y) = \left\{\begin{matrix} A_1 & \text{if } x \in A_1 \\ A_2 & \text{if } x \in A_2 \\ \{y\} & \text{if } x \notin A_1 \text{ and } x \notin A_2 \end{matrix}\right\}.$$

But $\partial A_1, \partial A_2$ are compact by hypothesis and □$\{y\}$ is also compact, so f is boundary- perfect, and thus A_1 or A_2 is countably compact by (b). Hence X is an almost semi weak CJ- space.

REFERENCES

1. *E. Michael, J-spaces, Topology and its Applications 102(2000) 315-339.*
2. *James R. Munkres, Topology A First Course, Prentice- Hall, 1974.*
3. *K. D. Joshi, Introduction to General Topology, New AGE International (p) Limited, Publishers, First Edition 1983, Reprint 2004.*
4. *T. Husain, Topology and Maps, Springer Science & Business Media, 2012.*
5. *S. C. Sharma, Topology Connectedness and Separation, Discovery Publishing House, First Published, 2006.*
6. *J. E. Vaughan, Jun- iti Nagata, K. P. Hart, Encyclopedia of General Topology, Elsevier, 2003.*